J Stat Phys (2010) 141: 60–67
DOI 10.1007/s10955-010-0026-5

Typical Recurrence for the Ehrenfest Wind-Tree Model

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Received: 20 April 2010 / Accepted: 9 July 2010 / Published online: 17 August 2010

Abstract We show that the typical wind-tree model, in the sense of Baire, is recurrent and has a dense set of periodic orbits. The recurrence result also holds for the Lorentz gas: the typical Lorentz gas, in the sense of Baire, is recurrent. These Lorentz gases need not be of finite horizon!

Keywords Ehrenfest wind-tree model · Lorentz gas · Recurrence · Translation surface · Square tiled surface

In 1912 Paul and Tatiana Ehrenfest proposed the wind-tree model of diffusion in order to study the statistical interpretation of the second law of thermodynamics and the applicability of the Boltzmann equation [3]. In the Ehrenfest wind-tree model, a point particle (the "wind") moves freely on the plane and collides with the usual law of geometric optics with randomly placed fixed square scatterers (the "trees"). The notion of "randomness" was not made precise, in fact it would have been impossible to do so before Kolmogorov laid the foundations of probability theory in the 1930s. We will call the subset of the plane obtained by removing the obstacles, the billiard table, and the motion of the point, the billiard flow.

The wind-tree model has been studied in [1, 4, 6] and the references therein. From the mathematical rigorous point of view, there have been two results on recurrence for wind-tree models, both on a periodic version where the scatterers are identical rectangular obstacles located periodically along a square lattice on the plane, one obstacle centered at each lattice point. Using results on skew products above rotations, Hardy and Weber [5] proved recurrence and abnormal diffusion of the billiard flow for special dimensions of the obstacles, for very special flow directions. More recently Hubert, Lelièvre, and Troubetzkoy have studied the general full occupancy periodic case [7]. They proved that, if the lengths of the sides of the rectangles belong to a certain dense G_δ subset $\mathcal{E}'$, then the dynamics is recurrent and

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they gave a lower bound on the diffusion rate. The recurrence was proven by analysis of the case when the lengths of the sides are rational and belong to the set

$$\mathcal{E} = \big\{(a,b) = (p/q, r/s) \in \mathbb{Q} \times \mathbb{Q} :$$
$$(p,q) = (r,s) = 1,\ 0 < p < q,\ 0 < r < s,$$
$$p, r \text{ odd}, q, s \text{ even}\big\}.$$

Another set of rationals lengths was used to prove the diffusion result. The set $\mathcal{E}' \subset (0,1)^2$ consists of dimensions which are sufficiently well approximable by both of these rational sets.

In this article we will prove the recurrence of typical (in a topological sense) wind-tree models. We consider the following model. Fix a finite or countable set of dimensions of obstacles $\mathcal{F} \subset (0,1)^2 \cup \{(0,0)\}$ such that $\mathcal{F} \cap (\mathcal{E} \cup \mathcal{E}') \neq \emptyset$. Let e be in this intersection, and let W_e denote the billiard table with an identical copy of the obstacle e centered at each lattice site. Consider the set of all wind-tree models $\mathcal{F}^{\mathbb{Z}^2}$ with the product topology on $\mathcal{F}^{\mathbb{Z}^2}$. A lattice site with an obstacle of dimension $(0,0)$ will be interpreted as a lattice site without obstacle. The recurrence of the table W_e will play a crucial role in our proof of recurrence of typical wind-tree models.

For each $W \in \mathcal{F}^{\mathbb{Z}^2}$, we will also refer to the corresponding table as W. More precisely, the table W is the plane $\mathbb{R}^2$ with the open rectangle of dimension $W_{i,j}$ centered at $(i,j) \in \mathbb{Z}^2$ removed for each $(i,j) \in \mathbb{Z}^2$. The phase space of the billiard flow Φ is the $W \times \mathbb{S}^1$, with the directional vector before and after collision with the boundary of the rectangles identified. There is a natural infinite invariant measure, the *phase volume* μ with $d\mu := dx\,dy\,d\theta$.

Consider a flow Φ on a measured topological space (Ω, μ) such that the measure of each open set is positive. A point $x \in \Omega$ is called recurrent for Φ, if for every neighborhood U of x, and any $T_0 > 0$, there is a time $T > T_0$ such that $\Phi_T(x) \in U$; the flow Φ itself is *recurrent* if μ-almost every point is recurrent. If μ is a finite measure, then the celebrated Poincaré recurrence theorem states that *for any U, μ almost every point of U is recurrent.*

In our setting, the billiard flow Φ is the constant unit speed flow, with elastic reflections with the rectangular obstacles (i.e. angle of incidence equals angle of reflection). This flow preserve the natural phase volume which is infinite, and thus one can not apply the Poincaré recurrence theorem. Our first result is that recurrence satisfies a 0-1 law:

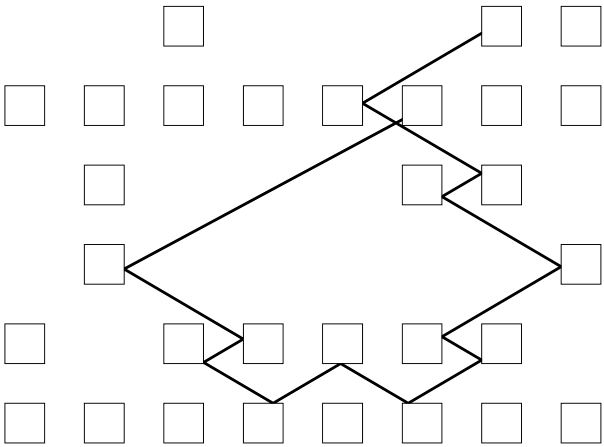
Proposition 1 *For each ergodic shift invariant measure on $\mathcal{F}^{\mathbb{Z}^2}$, the wind-tree models in $\mathcal{F}^{\mathbb{Z}^2}$ are almost surely recurrent, or almost surely non recurrent.*

The following topological result gives evidence that wind-tree models are almost surely recurrent.

Theorem 2 *There is a dense G_δ subset G of $\mathcal{F}^{\mathbb{Z}^2}$, such that the billiard flow is recurrent for every billiard table in G with respect to the natural phase volume.*

An orbit is called *regular* if it never hits the corner of an obstacle. A direction $\theta \in \mathbb{S}^1$ is called *purely periodic* if all regular orbits in this direction are periodic. The tables in $\mathcal{F}^{\mathbb{Z}^2}$ are called *square tiled* if $\mathcal{F}$ is a finite subset of $\mathcal{E} \cup \{(0,0)\}$. In this case there is a positive integer Q, the least common multiple of the denominators of the dimensions of the obstacles, such that each table $W \in \mathcal{F}^{\mathbb{Z}^2}$ can be tiled in the standard way (checkerboard tiling) by squares with side length $1/Q$.

Fig. 1 A random wind-tree model with obstacles of sizes $(1/2, 1/2)$ and $(0, 0)$



Theorem 3 *If $\mathcal{F}^{\mathbb{Z}^2}$ is square tiled, then there is a dense G_δ subset G of $\mathcal{F}^{\mathbb{Z}^2}$ such that the billiard flow is recurrent for every billiard table in G with respect to the natural phase volume. Furthermore for every billiard table in G there is a dense set of purely periodic directions $\theta \subset \mathbb{S}^1$.*

In the proofs we will prove the recurrence of certain first return maps. For this purpose, consider a map F on a measured topological space (Ω, μ) such that the measure of each open set is positive. A point $x \in \Omega$ is called recurrent for F if for every neighborhood U of x, there is a time $n > 0$ such that $F^n(x) \in U$; the map F itself is *recurrent* if μ-almost every point is recurrent.

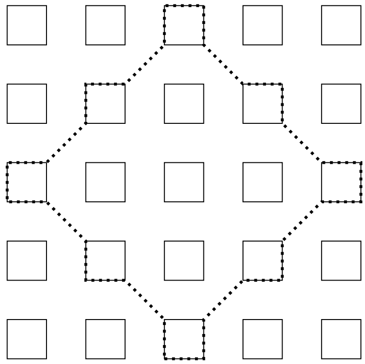
Next we introduce the first return maps we will use. Fix a table $W \in \mathcal{F}^{\mathbb{Z}^2}$ and suppose that $N \geq 1$. Let $B_N := \{(i, j) \in \mathbb{Z}^2 : |i| + |j| < N\}$ and $A_{N,N_1} := \{(i, j) \in \mathbb{Z}^2 : N \leq |i| + |j| < N_1\}$. Consider the continuous simple curve $D_N = D_N(W)$ in the billiard table W consisting of the segments of $|x| + |y| = N$ which are in the interior of the table (not in the obstacles) and the "outer" part of the boundary of the obstacles e with centers (i, j) satisfying $|i| + |j| = N$ (see Fig. 1). This curve separates the table into two parts, the finite (or inner) part and the infinite (or outer) part. Let $\hat{D}_N^-$ consist of the unit vectors with base point in D_N pointing into the finite part and $\hat{D}_N^+$ pointing into the infinite part of the table. Let $\hat{D}_N := \hat{D}_N^- \cup \hat{D}_N^+$ and $\hat{D} = \bigcup_{N \geq 1} \hat{D}_N$. Consider the first return maps $f_N : \hat{D}_N \to \hat{D}_N$ (wherever they are well defined) and $f : \hat{D} \to \hat{D}$. Let dl be the length measure on $\hat{D}$. We will refer to the measure $d\nu := dl \times d\theta$ as the *phase area*. Clearly f preserves the phase area. Note that the singular points (points whose images, or preimages, hit a corner or are tangent to a side of an obstacle) are of measure 0. Also note that if we consider that map f restricted to the set $\hat{D}_N^-$, then almost every point must return to $\hat{D}_N$ since the measure space $\bigcup_{n=1}^N \hat{D}_n$ is a finite measure space and the only possibility to escape this finite part is by passing through $\hat{D}_N$.

The proofs of the results rely on the following lemma.

Lemma 4 *The following statements are equivalent.*

(1) *The wind-tree model W is recurrent*
(2) *$f : \hat{D} \to \hat{D}$ is recurrent*

Fig. 2 The curve D_2 as seen on the table W_e consisting of the obstacle e at such lattice point



(3) *there is a positive sequence* $\varepsilon_n \searrow 0$ *and a sequence* $N_n \to \infty$ *such that* f_{N_n} *is well defined for at least* $(1-\varepsilon_n)\%$ *of the points in* $\hat{D}_{N_n}$.

Proof Clearly (1) implies (2) and (2) implies (3). We will now show that (3) implies (2). We claim that if $N_1 < N_2$, and if f_{N_2} is well defined almost everywhere, then f_{N_1} is also well defined almost everywhere. Simply consider the map f_{N_1} induced on the set $\bigcup_{N \le N_2} \hat{D}_N$. This map is well defined almost everywhere since f_{N_2} is. Thus f_{N_1} is recurrent by the Poincare recurrence theorem. Thus to show that each f_N is actually defined almost everywhere and thus f is recurrent, it suffices to show that f_N is, for infinitely many N.

Note that the map f is invertible. Consider a set $U \subset \hat{D}_{N_n}$ which never recurs to $\hat{D}_{N_n}$. We claim that $f^j U \cap f^k U = \emptyset$ for all $j > k \ge 0$. If not then by the invertibility of f we would have $f^{j-k} U \cap U \ne \emptyset$, i.e. some points in U recur to $U \subset \hat{D}_{N_n}$, a contradiction. This implies that since the set $\hat{D}_{N_n}$ are of finite measure, almost every point in U can visit each set $\hat{D}_{N_m}$ only a finite number of times. Thus we can define for almost every $x \in U$ a (finite time) $m_n(x) \ge 0$ be the last time the orbit of x visits $\hat{D}_{N_m}$. The map $F(x) = f^{m_n(x)}(x)$ is a measure preserving map of U into the set of nonrecurrent points in $\hat{D}_{N_m}$. This set has measure at most ε_m. Since this hold for all $m \ge n$, the set U is of measure 0, i.e. f_N is well defined almost everywhere, and thus f is recurrent.

Finally we need to show that (2) implies (1). Consider any small open ball B in the phase space. Flow each (non-singular) point in this ball until it hits the set $\hat{D}$. Since the ball is open, it has positive phase volume, and its image on the set $\hat{D}$ also has positive phase area. Almost every of these points is f recurrent.

Fix a nonsingular $x \in B$ such that $x_N := \Phi_t(x) \in \hat{D}_N$. Note that by transversality and the Fubini theorem almost every $x \in B$ corresponds to a f-recurrent x_N. To conclude the proof we suppose that x_N is f-recurrent and we will show that this implies that x is Φ-recurrent. Choose a open neighborhood U of x small enough that for each $y \in U$ there is a $t(y)$ very close to t such that $\Phi_{t(y)}(y) \in \hat{D}_N$. Let $U' = \{\Phi_{t(y)}(y) : y \in U\}$. This is a small neighborhood of $\Phi_t(x)$, and by the above results there is an (arbitrarily large) n such that $f^n x_N \in U'$. Thus $f^n x_N = \Phi_s(x_N) = \Phi_{s+t}(x) \in U'$ for some large s. Since this point $\Phi_{s+t}(x)$ is in U' is the image $\Phi_{t(y_0)} y_0$ of some $y_0 \in U$. Thus $\Phi_{s+t-t(y_0)}(x) = y \in U$ and we conclude that x is recurrent. □

Proof of Theorem 2 The idea of the proof is simple. The table W_e is recurrent. A table in our dense G_δ will have infinitely many large annuli for which it agrees with W_e, i.e. it has the obstacle e at all lattice sites in the annuli. The widths of these annuli will increase sufficiently quickly to guarantee the recurrence.

Fix $\varepsilon > 0$ and $N \geq 1$. Fix a cylinder set in $C = C_{k,N} \in \mathcal{F}^{B_N}$, i.e. C is given by specifying the rectangle size (or absence of rectangle) at all the lattice points in B_N (the index k enumerates the finite (or countable) collection of all such cylinder sets). Let $N_1 \gg N$. Consider the cylinder set $C' = C'_{k,N,N_1}$ such that $C' \subset C$, and such that for each $c \in C'$, $c_{(i,j)} = e$ for all $(i, j) \in A_{N,N_1}$. Consider the table W_e. Since it is recurrent, for each fixed N, we can choose $N_1 = N_1(N, \varepsilon)$ sufficiently large so that on this table $(1 - \varepsilon)\%$ of the points in $\hat{D}_N^+$ recur to $\hat{D}_N$ before leaving $A_{N,N1}$. The dynamics for any table in the cylinder C' is identical to the dynamics on the table W_e as long as it stays in the annulus A_{N,N_1}. The $(1 - \varepsilon)\%$-recurrence hold for all the tables in the cylinder C' since the wind-tree tables in the cylinder agree with W_e on A_{N,N_1}.

Consider the set $O_\varepsilon := \bigcup_{N\geq 1} \bigcup_k C'_{k,N,N_1(N,\varepsilon)}$. Since cylinder sets are open, this set is open. It is dense since the union is taken over all cylinder sets. Now fix a sequence $\varepsilon_n \searrow 0$, and let $G := \bigcap_n O_{\varepsilon_n}$. Clearly G is a dense G_δ set. For each table $W \in G$ and for all n there exists $N_n = N_n(W)$ and $k(N_n)$ such that $W \in C'_{k(N_n),N_n,N_1(N_n,\varepsilon_n)}$. This means that for each $n \geq 1$ at least $(1 - \varepsilon_n)\%$ of the points in the set $\hat{D}_{N_n}$ recur to $\hat{D}_{N_n}$. We apply Lemma 4 to conclude the recurrence. □

For the proof it is very important that in the annuli the tables agree with a recurrent full occupancy table. On the other hand for the recurrence the shape of the table in between the annuli is not at all important. Instead of taking rectangular scatterers, we could choose circular scatterers, or more byzantine ones (as long as we can define billiard dynamics preserving the phase volume, for example if they are piecewise C^1), disjoint and finitely many in any compact region. Furthermore, in the annuli it is not necessary that all the obstacles have the same $e \in \mathcal{E}$, on can fix a sequence $\{e_n\} \subset \mathcal{E}$, and use the e_n in the nth annulus.

We can also relax the fact that the tables agree with a recurrent full occupancy table on the annuli. We can replace this by an almost agreement in the following sense. The centers of the obstacles could be assumed to be uniformly distributed in a small open ball around each lattice point. For a cylinder set in which the restriction to an annulus (or another finite set) the centers are very close to the lattice, the points in $\hat{D}_N^+$ are almost recurrent. Such perturbations will change the "itinerary" of only a small set of finite time trajectories, but the identification of measures spaces and notions of smallness are somewhat cumbersome to keep track of.

Proof of Theorem 3 The recurrence is a special case of Theorem 2. We will impose addition constraints on the sequence of annuli to conclude the denseness of purely periodic directions.

Consider a periodic orbit. Any sufficiently close parallel orbit will also be periodic, hitting the same sequence of sides and having the same geometric length. Geometrically this set of parallel periodic orbits, which we will call a *periodic strip*, consists of the parallel orbit of an open interval of initial points, with the endpoints having singular orbits. A periodic strip is often called a cylinder in the translation surface literature, but we will not do so here to avoid confusion with the cylinder set topology on $\mathcal{F}^{\mathbb{Z}^2}$. A purely periodic direction is called *strongly parabolic* if the phase space decomposes into an infinite number of periodic strips isometric to each other. In [7] it was shown that the set of strongly parabolic directions θ for the table W_e are dense in $\mathbb{S}^1$. These directions have rational slope ($\pm p/q$ with p and q depending on θ). We will show that for any strongly parabolic direction θ for W_e, for any table $W \in G$, a.e. orbit on this table in the direction θ is periodic.

Fix θ a strongly parabolic direction for W_e, and let M be the common (geometric) length of the periodic strips. Let n be so large that $N_1(N_n, \varepsilon_n) - N_n > 2M$ and let $P_n := (N_n + N_1(N_n, \varepsilon_n))/2$. On the table W_e all the periodic strips crossing D_{P_n} stay in $A_{N_n, N_1(N_n,\varepsilon_n)}$. All regular orbits in W staring on D_{P_n} are periodic since the table W coincides with W_e on this set.

Now consider any phase point in W with regular orbit starting strictly inside B_{P_n}. First of all since D_{P_n} consists completely of periodic orbits this orbit can not reach D_{P_n} without being one of these periodic orbits. If the orbit does not reach D_{P_n} then it stays completely inside B_{P_n}. Since the orbit's slope is rational, bounded and the billiard table is square tiled with squares of side length $1/Q$, the y coordinate can only take a finite number of values when crossing the segments $\{x = m/Q\} \cap B_{P_n}$ with $m \in \mathbb{Z}$. Thus it must visit some point $(m_0/Q, y_0)$ twice with the same direction. The orbit is periodic since the dynamics is invertible. $\square$

Proof of Proposition 1 We use Lemma 4, the wind-tree model $W \in \mathcal{F}^{\mathbb{Z}^2}$ is recurrent if and only condition (3) of the lemma holds: there are sequences $\varepsilon_n = \varepsilon_n(W) \searrow 0$ and $N_n = N_n(W) \to \infty$, such that $(1 - \varepsilon_n)\%$ of all points in $\hat{D}_{N_n}$ recur to $\hat{D}_{N_n}$.

We claim that a stronger property is true: fix a sequence $\varepsilon'_n \searrow 0$ (this sequence does not depend on the table), a wind-tree model is recurrent iff there exists $M_n = M_n(W) > N_n = N_n(W) \geq n$ such that $(1 - 2\varepsilon'_n)\%$ of all points in $\hat{D}_{N_n}$ recur to $\hat{D}_{N_n}$ before hitting $\hat{D}_{M_n}$. Clearly this condition implies condition (3) of Lemma 4. To see the converse fix a sequence $\varepsilon'_n \searrow 0$. If a wind-tree model satisfies condition (3), then we can choose $M_n > N_n$ such that $(1-2\varepsilon_n)\%$ of all points in $\hat{D}_{N_n}$ recur to $\hat{D}_{N_n}$ before hitting $\hat{D}_{M_n}$. By choosing a subsequence we can suppose that $N_n \geq n$. Then the recurrent table satisfies the above condition for the sequence $\{\varepsilon'_n\}$ since if $\varepsilon_n \to 0$ faster than ε'_n, then it satisfies the condition with ε'_n, and if not then we can choose a subsequence for which it goes faster.

One needs to specify only a finite part of the table to check this property at a fixed stage n, i.e. if a table $W \in \mathcal{F}^{\mathbb{Z}^2}$ is recurrent, then all the tables in the cylinder set with the obstacles specified to be those of W at lattice points (i, j) with $|i| + |j| \leq M_n$ satisfy this property for this fixed n. Let O_{N_n,M_n} denote the union of all cylinder sets such that this happens at stage n. This is an open set. Thus by the above characterization, the set of recurrent wind-tree models can be written as $\bigcap_{n=1}^{\infty} \bigcup_{M_n > N_n \geq n} O_{N_n,M_n}$, thus it is a Baire measurable set.

The notion of recurrence is shift invariant (in the space $\mathcal{F}^{\mathbb{Z}^2}$). Thus since the set of recurrent wind-tree models is measurable and invariant, it is of measure 0 or 1 for any invariant measure which is ergodic for the $\mathbb{Z}^2$ shift. $\square$

1 Lorentz Gas

A Lorentz gas is similar to the Ehrenfest wind-tree model, with the rectangular obstacles replaced by strictly convex (C^3) obstacles. A Lorentz gas is said to verify the *finite horizon* condition if any infinite line intersects infinitely many obstacles with bounded gaps between the intersections. In some of the literature, the word finite horizon also assumes that the minimal distance between obstacles is strictly positive [8, 9], we will call this *finite horizon**. The long standing conjecture that finite horizon* periodic Lorentz gases are recurrent was independently resolved by Conze [2] and Schmidt [10] in the 1990's, building on previous results on the hyperbolic structure. Using the hyperbolic structure models, Lenci has shown that Baire typical Lorentz gases with finite horizon* are recurrent [9]. Here we prove

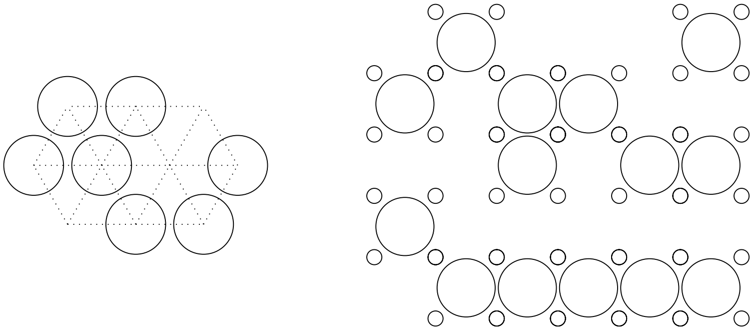


Fig. 3 Lorentz gases with triangular and square lattices

that Baire typical Lorentz gases are recurrent. Our typical gas will satisfy a weaker property which we call *locally finite horizon*: (i) the distance between obstacles is still strictly bounded away from 0; and (ii) any infinite line must intersect infinitely many obstacles. The reason that (ii) holds is that any infinite line must intersect infinitely many of the annuli constructed in the proof of the theorem, and in these annuli the billiard table agrees with a finite horizon model, thus each line must intersect infinitely many obstacles. However the gaps between the intersections are not necessarily bounded since we do not control the behavior of the billiard table between the annuli.

Rather than try to state a general result, we give two examples. Generalizations to other situations should be clear.

(1) Let $\mathbb{T}$ denote the triangular lattice. Fix e a convex open set with C^3 boundary (for example a ball), sufficiently large so that if we place the obstacle e at each lattice site, then the corresponding infinite table satisfies the finite horizon* condition. Let 0 denote the absence of an obstacle. Then the set of Lorentz billiard tables we consider is $\{0, e\}^{\mathbb{T}}$, note that since we are allowing empty cells, these tables do not necessarily have finite horizon, not even locally finite horizon (for example the table with no obstacles).

(2) Consider the $\mathbb{Z}^2$ lattice. Here we let e denote the obstacle consisting of the union of 5 convex open sets with C^3 boundary (again for examples balls). The 5 convex sets are chosen so the table consisting of the obstacle e at each lattice site is of finite horizon. Again 0 denotes the absence of an obstacle. We consider the set of Lorentz gases $\{0, e\}^{\mathbb{Z}^2}$. These tables, like those above, do not necessarily have finite horizon, nor locally finite horizon.

The proof of the following theorem is essentially identical to the proof of Theorem 2 and will be omitted.

Theorem 5 *There is a dense G_δ subset G of each of the above two examples, such that the billiard flow is recurrent for every billiard table in G with respect to the natural phase volume.*

All the tables in the dense G_δ will have a locally finite horizon and some will not have finite horizon. We can even construct a dense G_δ of tables with locally finite horizon, all of which do not satisfy the finite horizon condition. This is done by additionally insuring that there are infinitely many increasing annuli without obstacles.

It would be interesting to investigate if this implies ergodicity like in the finite horizon case (see [8, 9]). It would also be interesting to get quantitive information on the recurrence properties of W_e, which would allow to construct explicit examples of recurrent Lorentz gases with infinite horizon.

Acknowledgements Many thanks to Marco Lenci and the referees for useful comments.

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